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A discrete approach to topological quantum field theories

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First, we describe a rather general scheme for constructing three-dimensional euclidean topological quantum field theories, whose basic building blocks are provided by the representation theory of a certain class of (bi-)algebras. Secondly, we discuss in some detail examples, where the algebra is either the function algebra of a finite group, the group algebra of a finite group or a deformation of the enveloping algebra of a classical simple Lie group.

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1 Introduction

The concept of topological quantum field theory [1,2] provides a useful framework for discussing the mutual interplay between quantum field theory on the one hand and geometry and topology on the other, and has led to important progress during the last years (see, e.g., refs. [3–6]). It is therefore of interest to provide tools for rigorous construction of such models. Because of scale invariance, topological QFTs are more tractable for rigorous approach than QFTs in general, whose construction can only be carried out with difficulties, even in two space–time dimensions. In this contribution we discuss a specific construction of a class of euclidean three-dimensional topological QFTs. It is based on representation theoretic data for certain associative algebras (or bialgebras), and generalizes the constructions proposed in refs. [7–9].

In order to fix notation let us briefly recall the defining properties of three-dimensional unitary topological quantum field theories (see ref. [2]).

In the following all manifolds M are assumed to be smooth, oriented, compact and three-dimensional, and all surfaces Σ are assumed to be smooth, oriented, compact and closed. The connected components of the boundary ∂M are denoted $\Sigma_1, \ldots, \Sigma_n$.

A topological QFT [2] associates with each smooth, oriented, compact, connected surface Σ a finite-dimensional complex Hilbert space V_Σ, and hence with M the Hilbert space

$$V_{\partial M} = V_{\Sigma_1} \otimes \cdots \otimes V_{\Sigma_n} . \tag{1.1}$$

The inner product on V_Σ will be denoted $\langle \cdot , \cdot \rangle_\Sigma$, and it is assumed that, if Σ^* denotes the surface Σ with opposite orientation, then

$$V_{\Sigma^*} = V_\Sigma^* , \tag{1.2}$$

where V_Σ^* denotes the dual space to V_Σ. In other words, with each Σ is associated a non-degenerate bilinear form $(\cdot , \cdot)_\Sigma : V_\Sigma : V_{\Sigma^*} \to \mathbb{C}$ such that

$$(x, y)_\Sigma = (y, x)_{\Sigma^*} , \quad x \in V_\Sigma, y \in V_{\Sigma^*} , \tag{1.3}$$

$$(x, y)_\Sigma = \langle x, y^* \rangle_\Sigma , \quad x \in V_\Sigma, y \in V_{\Sigma^*} , \tag{1.4}$$

$$x^{**} = x , \quad x \in V_\Sigma , \tag{1.5}$$

where $(\cdot , \cdot)_\Sigma$ defines the identification (1.2), and the map $y \to y^*$ from V_{Σ^*} to V_Σ is the composition of this identification map and the canonical antilinear isomorphism between V_Σ and V_Σ^*.

Next, to each orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ between surfaces Σ and Σ' is associated an isomorphism $U(f) : V_\Sigma \to V_{\Sigma'}$, in such a way that

$$U(f_1 \circ f_2) = U(f_1) U(f_2) \tag{1.6}$$

for any pair $f_2: \Sigma \to \Sigma'$ and $f_1 : \Sigma' \to \Sigma''$ of diffeomorphisms.

And, finally, to each manifold M is associated a unique vector $Z(M) \in V_{\partial M}$ such that the following conditions hold:

(i) If $F: M \to M'$ is an orientation preserving diffeomorphism between manifolds M and M' and we set $f = F|_{\partial M} : \partial M \to \partial M'$, then

$$Z(M') = U(f) Z(M) . \tag{1.7}$$

(ii) If M_1 and M_2 are two manifolds whose boundaries share a (not necessarily connected) surface Σ, i.e., $\partial M = \Sigma_1 \cup \Sigma$ and $\partial M_2 = \Sigma_2 \cup \Sigma^*$, and if $M_1 \sqcup_\Sigma M_2$ denotes the manifold obtained by identifying the two copies of Σ pointwise, then

$$Z(M_1 \sqcup_\Sigma M_2) = (Z(M_1), Z(M_2))_\Sigma , \tag{1.8}$$

where the right-hand side denotes the vector in $V_{\Sigma_1} \otimes V_{\Sigma_2}$ obtained by contracting $Z(M_1)$ and $Z(M_2)$ with respect to V_Σ, i.e., if

$$Z(M_1) = \sum_i x_i^1 \otimes y_i^1 , \qquad Z(M_2) = \sum_i x_j^2 \otimes y_j^2 ,$$

where $x_i^1 \in V_{\Sigma_1}$, $x_j^2 \in V_{\Sigma_2}$, $y_i^1 \in V_\Sigma$, $y_j^2 \in V_{\Sigma^*}$, then

$$(Z(M_1), Z(M_2))_\Sigma = \sum_{i,j} (y_i^1, y_j^2)_\Sigma x_i^1 \otimes x_j^2 .$$

(iii) If M is a manifold and M^* denotes M with opposite orientation, then

$$Z(M^*) = Z(M)^* . \tag{1.9}$$

(iv) If Σ is a surface, then

$$Z(\Sigma \times [0, 1]) = 1_{V_\Sigma} \tag{1.10}$$

as an element in $V_\Sigma^* \otimes V_\Sigma = \mathrm{End}(V_\Sigma)$.

(v) Finally,

$$V_\phi = \mathbb{C} , \tag{1.11}$$

which, in particular, implies that, if M is a closed manifold, then $Z(M) \in \mathbb{C}$.

In the following we exhibit a class of models fulfilling (i)–(v) above. The associative $*$-algebras, denoted by $\mathfrak{A}$, underlying the construction are introduced in section 2 together with a finite set I of irreducible $*$-representations of $\mathfrak{A}$ realized on Hilbert spaces fulfilling a certain set of conditions (1)–(5), and it is shown how a construction of a topological quantum field theory based solely on these assumptions can be accomplished.

In sections 3 and 4 we discuss in detail some concrete cases, namely the case of finite groups in section 3 and the case where the algebra $\mathfrak{A}$ is a deformation of the enveloping algebra of a simple classical Lie algebra (quantum group) in section 4.

2. General construction of models

2.1. ALGEBRAS AND REPRESENTATIONS

We assume in the following that $\mathfrak{A}$ is an associative algebra over the complex numbers with an antilinear involution $a \to a^*$, and that we are given a *finite* set I of irreducible $*$-representations of $\mathfrak{A}$ on Hilbert spaces H_i, $i \in I$. In the following all representations of $\mathfrak{A}$ will be assumed to be $*$-representations on Hilbert spaces. We assume that there is a notion of "tensor" product of representations of $\mathfrak{A}$, which associates to any pair π, ρ of representations from a class $\mathscr{C}$ containing I, a representation $\pi \underline{\otimes} \rho \in \mathscr{C}$ on a Hilbert space $H_{\pi \underline{\otimes} \rho}$, and which fulfills the following properties (see ref. [10]):

(1) *Associativity*. If π, ρ and η are representations in $\mathscr{C}$ then

$$(\pi \underline{\otimes} \rho) \underline{\otimes} \eta = \pi \underline{\otimes} (\rho \underline{\otimes} \eta) . \tag{2.1}$$

(2) *Reducibility*. For $i, j \in I$ the tensor product $i \underline{\otimes} j$ can be decomposed into a finite direct sum of representations from I. Thus, if V_{ij}^k denotes the space of intertwiners between $i \underline{\otimes} j$ and k, $k \in I$, i.e. linear mappings $\alpha : H_k \to H_{i \underline{\otimes} j}$ such that

$$i \underline{\otimes} j(a) \circ \alpha = \alpha \circ k(a) , \quad a \in \mathfrak{A} , \tag{2.2}$$

then

$$N_{ij}^k \equiv \dim V_{ij}^k < \infty \tag{2.3}$$

and

$$H_{i \underline{\otimes} j} = \bigoplus_{k \in I} V_{ij}^k \otimes H_k \tag{2.4}$$

as $\mathfrak{A}$-modules, where the identification map in (2.4) is given by

$$\alpha \otimes v \to \alpha(v) , \quad \alpha \in V_{ij}^k, v \in H_k . \tag{2.5}$$

Note that N_{ij}^k is the multiplicity of k in $i \underline{\otimes} j$, and that V_{ij}^k is a Hilbert space with inner product $\langle \cdot , \cdot \rangle$ given by

$$\langle \alpha, \beta \rangle 1_k = \beta^* \circ \alpha , \tag{2.6}$$

which is well defined since $\beta^* \circ \alpha$ commutes with $k(\mathfrak{A})$, and hence is proportional to the identity operator 1_k on H_k, since k is irreducible.

(3) *Involution*.

(i) There exists a distinguished representation $0 \in I$ such that $0 \otimes i \simeq i \otimes 0 \simeq i$ for $i \in I$, where $\simeq$ denotes unitary equivalence.

(ii) For each $i \in I$ there exists a unique $i^\vee \in I$ such that $N_{ii^\vee}^0 = 1$ and $N_{ij}^0 = 0$ for $j \neq i^\vee$.

(iii) $(i^\vee)^\vee = i$ for $i \in I$, i.e., $N_{i^\vee i}^0 = 1$ and $N_{i^\vee j}^0 = 0$ for $j \neq i$.

(4) *Tensor product of intertwiners*. Given representations π, π', ρ, ρ' of $\mathfrak{A}$ from $\mathscr{C}$ and intertwiners $u : H_\pi \to H_{\pi'}$ and $v : H_\rho \to H_{\rho'}$ there is a canonical intertwiner

$$u \underline{\otimes} v : H_{\pi \underline{\otimes} \rho} \to H_{\pi' \underline{\otimes} \rho'} \tag{2.7}$$

depending bilinearly on u and v and fulfilling

$$(u' \underline{\otimes} v')(u \underline{\otimes} v) = (u'u) \underline{\otimes} (v'v) , \tag{2.8}$$

$$1_\pi \underline{\otimes} 1_\rho = 1_{\pi \underline{\otimes} \rho} , \tag{2.9}$$

$$(u \underline{\otimes} v) \underline{\otimes} w = u \underline{\otimes} (v \underline{\otimes} w) , \tag{2.10}$$

$$(u \underline{\otimes} v)^* = u^* \underline{\otimes} v^* , \tag{2.11}$$

for any intertwiners u, v, u', v', w between appropriate representations in $\mathscr{C}$.

We shall need some additional assumptions of a more technical nature. In order to formulate these, we fix once and for all intertwiners

$$\psi_k \in V_{kk^\vee}^0 , \qquad \varphi_i \in V_{0i}^i , \qquad \varphi \in V_{i0}^i , \tag{2.12}$$

which by (3) are unique up to multiplicative constants. Fixing ψ_k to be a partial isometry and φ_i and ${}_i\varphi$ to be unitary, they are determined up to phases, and we assume these can be chosen such that $\varphi_0 = {}_0\varphi = \psi_0$ and the following conditions are fulfilled.

(5) *Technical assumptions.*

(i)
$$1_i \underline{\otimes} \varphi_j = {}_i\varphi \underline{\otimes} 1_j \,, \tag{2.13}$$

(ii)
$$\varphi_k \circ (1_0 \underline{\otimes} \alpha) = \alpha \circ (\varphi_i \underline{\otimes} 1_j) \,, \tag{2.14}$$

(iii)
$$_k\varphi \circ (\alpha \underline{\otimes} 1_0) = \alpha \circ (1_i \underline{\otimes}_j \varphi) \,, \tag{2.15}$$

(iv)
$$\begin{aligned} &\psi_k \circ (\alpha \underline{\otimes} \beta) \circ (1_i \underline{\otimes} \psi_j^* \underline{\otimes} 1_{i^\vee}) \circ (1_i \underline{\otimes} \varphi_{i^\vee}^*) \circ \psi_i^* \\ &= \psi_{k^\vee} \circ (\beta \underline{\otimes} \alpha) \circ (1_{j^\vee} \underline{\otimes} \psi_{i^\vee}^* \underline{\otimes} 1_j) \circ (1_{j^\vee} \underline{\otimes} \varphi_j^*) \circ \psi_{j^\vee}^* \,, \end{aligned} \tag{2.16}$$

for all $\alpha \in V_{ij}^k$, $\beta \in V_{j^\vee i^\vee}^{k^\vee}$,

(v)
$$F_i \in \mathbb{R} \setminus \{0\} \quad \text{for } i \in I \,, \tag{2.17}$$

where F_i is defined by

$$_i\varphi \circ (1_i \underline{\otimes} \psi_{i^\vee}) \circ (\psi_i^* \underline{\otimes} 1_i) \circ \varphi_i^* = F_i 1_i \,. \tag{2.18}$$

Note that the left-hand side of (2.18) commutes with $i(\mathfrak{A})$ and hence is proportional to 1_i.

This finishes the specification of our assumptions.

Remark 2.1. The associativity conditions (2.1) and (2.10) may be relaxed to the following *quasi-associativity* conditions.

To each triple (π_1, π_2, π_3) of representations from $\mathscr{C}$ is associated a unitary intertwiner

$$\phi(\pi_1, \pi_2, \pi_3) : H_{(\pi_1 \underline{\otimes} \pi_2) \underline{\otimes} \pi_3} \to H_{\pi_1 \underline{\otimes} (\pi_2 \underline{\otimes} \pi_3)} \,,$$

fulfilling the *pentagon identity*

$$\begin{aligned} &(1 \underline{\otimes} \phi(\pi_2, \pi_3, \pi_4)) \circ \phi(\pi_1, \pi_2 \underline{\otimes} \pi_3, \pi_4) \circ (\phi(\pi_1, \pi_2, \pi_3) \underline{\otimes} 1) \\ &= \phi(\pi_1, \pi_2, \pi_3 \underline{\otimes} \pi_4) \circ \phi(\pi_1 \underline{\otimes} \pi_2, \pi_3, \pi_4) \end{aligned} \tag{2.1'}$$

as operators from $H_{((\pi_1 \underline{\otimes} \pi_2) \underline{\otimes} \pi_3) \otimes \pi_4}$ to $H_{\pi_1 \underline{\otimes} (\pi_2 \underline{\otimes} (\pi_3 \otimes \pi_4))}$, for any $\pi_1, \pi_2, \pi_3, \pi_4 \in \mathscr{C}$. Moreover, if $u_i : H_{\pi_i} \to H_{\rho_i}$, $i = 1, 2, 3$, are intertwiners between representations from $\mathscr{C}$ then

$$\phi(\rho_1, \rho_2, \rho_3) \circ (u_1 \underline{\otimes} u_2) \underline{\otimes} u_3 = u_1 \underline{\otimes} (u_2 \underline{\otimes} u_3) \circ \phi(\pi_1, \pi_2, \pi_3) \tag{2.10'}$$

replaces condition (2.10).

Let now P_n denote the set of possible ways of forming an n-fold tensor product of n representations in a given order, i.e. the set of configurations of parentheses defining a tensor product of n representations. Given $\pi_1, \ldots, \pi_n \in \mathscr{C}$, denote by π_p the tensor product of $\pi_1, \ldots, \pi_n$ corresponding to $p \in P_n$, and by H_{π_p} the Hilbert space on which it acts. A version of MacLane's coherence theorem [11] implies that ϕ defines for each pair $p, q \in P_n$ a unique intertwiner $\phi_{p,q}: H_{\pi_q} \to H_{\pi_p}$ such that

$$\phi_{p,q} \circ \phi_{q,r} = \phi_{p,r} \tag{2.19}$$

for $p, q, r \in P_n$, and by (2.8), (2.9), (2.11) each $\phi_{p,q}$ is unitary.

Furthermore, if $u_i: H_{\pi_i} \to H_{\rho_i}$, $i = 1, \ldots, n$, are intertwiners between representations from $\mathscr{C}$, we denote by $u_p: H_{\pi_p} \to H_{\rho_p}$ the tensor product of $u_1, \ldots, u_n$ corresponding to $p \in P_n$. As a consequence of (2.10′) and the definition of $\phi_{p,q}$ it follows that

$$\phi_{p,q} \circ u_q = u_p \circ \phi_{p,q} \tag{2.20}$$

for $p, q \in P_n$.

Using the $\phi_{p,q}$ to keep track of parentheses in tensor products all results below in this section still hold within this generalized setting provided the technical assumptions (5i)–(5v) are modified accordingly:

(i′)
$$(1_i \underline{\otimes} \varphi_j) \circ \phi(i, 0, j) = {}_i\varphi \underline{\otimes} 1_j , \tag{2.13′}$$

(ii′)
$$\varphi_k \circ (1_0 \underline{\otimes} \alpha) \circ \phi(0, i, j) = \alpha \circ (\varphi_i \underline{\otimes} 1_j) , \tag{2.14′}$$

(iii′)
$$_k\varphi \circ (\alpha \underline{\otimes} 1_0) \circ \phi(i, j, 0) = \alpha \circ (1_i \underline{\otimes}_j \varphi)' , \tag{2.15′}$$

(iv′)
$$\begin{aligned} &\psi_k \circ (\alpha \underline{\otimes} \beta) \circ \phi^{-1}(i, j, j^\vee \underline{\otimes} i^\vee) \circ (1_i \underline{\otimes} \phi(j, j^\vee, i^\vee)) \\ &\quad \circ (1_i \underline{\otimes} (\psi_j^* \underline{\otimes} 1_{i^\vee})) \circ (1_i \underline{\otimes} \varphi_{i^\vee}^*) \circ \psi_i^* \\ &= \psi_{k^\vee} \circ (\beta \underline{\otimes} \alpha) \circ \phi^{-1}(j^\vee, i^\vee, i \underline{\otimes} j) \circ (1_{j^\vee} \underline{\otimes} \phi(i^\vee, i, j)) \\ &\quad \circ (1_{j^\vee} \underline{\otimes} (\psi_{i^\vee}^* \underline{\otimes} 1_i)) \circ (1_{j^\vee} \underline{\otimes} \varphi_j^*) \circ \psi_{j^\vee}^* , \end{aligned} \tag{2.16′}$$

(v′)
$$F_i \in \mathbb{R} \setminus \{0\} \quad \text{for } i \in I , \tag{2.17′}$$

where F_i is defined by

$$_i\varphi \circ (1_i \underline{\otimes} \psi_{i^\vee}) \circ \phi(i, i^\vee, i) \circ (\psi_i^* \underline{\otimes} 1_i) \circ \varphi_i^* = F_i 1_i .$$

In the following we shall, however, mostly stick to the associative setting for notational convenience.

2.2. VECTOR SPACES

We are now in position to define the vector space $V_{(\Sigma, \mathscr{S})}$ associated to a surface Σ equipped with a triangulation $\mathscr{S}$. First decorate each link in the triangulation by an arrow and a label from I, in an arbitrary way. Then consider a triangle t in $\mathscr{S}$ and assume that the labels attached to its links are i, j, k in cyclic order with respect to the orientation of t inherited from Σ. If all three arrows on the links in t point in positive direction we associate with the labeled triangle t the vector space

$$V_t = V_{ij}^{k^\vee} \tag{2.21}$$

and adopt the convention that switching an arrow on a link is compensated by applying the involution $i \to i^\vee$ to the corresponding label. This defines V_t for any configuration of arrows and labels on the links in t. Clearly, the definition (2.21) only makes sense if the right-hand side only depends on the cyclic ordering of i, j, k, i.e., we have to provide canonical isomorphisms

$$V_{ij}^{k^\vee} \simeq V_{ki}^{j^\vee} \simeq V_{jk}^{i^\vee} . \tag{2.22}$$

This is done as follows. Given $\alpha \in V_{ij}^{k^\vee}$, there is a unique $\tilde{\alpha} \in V_{ki}^{j^\vee}$, such that

$$\psi_k \circ (1_k \underline{\otimes} \alpha) = \psi_{j^\vee} \circ (\tilde{\alpha} \underline{\otimes} 1_j) . \tag{2.23}$$

In fact, an explicit formula for $\tilde{\alpha}$ as well as for the inverse mapping $\beta \to \underset{\sim}{\beta}$ of $\alpha \to \tilde{\alpha}$ can be written down (see ref. [10]). Thus $\alpha \to \tilde{\alpha}$ defines an isomorphism $V_{ij}^{k^\vee} \simeq V_{ki}^{j^\vee}$. Moreover, it follows from assumption (5iv) above that

$$\tilde{\tilde{\tilde{\alpha}}} = \alpha , \quad \alpha \in V_{ij}^{k^\vee} \tag{2.24}$$

(see ref. [10]), which implies that the isomorphisms $\alpha \to \tilde{\alpha}$ yield a consistent identification of the spaces in (2.22) as desired.

We then set

$$V_{(\Sigma,\mathscr{S})} = \bigoplus_{(i_l)} \bigotimes_{t \in \mathscr{S}} V_t , \tag{2.25}$$

where the tensor product is over all triangles in $\mathscr{S}$ and the direct sum is over all labelings of the links l in $\mathscr{S}$ by labels $i_l \in I$, whereas the configuration of arrows on links is fixed. By our convention relating flips of arrows to the involution $i \to i^\vee$, it follows that $V_{(\Sigma,\mathscr{S})}$ is independent of the choice of arrows. Moreover, it is an easy consequence of definition (2.23) and the fact that ψ_k is a partial isometry, that the mappings $\alpha \to \tilde{\alpha}$ are unitary with respect to the natural inner products given by (2.6), and hence these define an inner product $\langle \cdot , \cdot \rangle_t$ on V_t under the identification (2.22). Thus, a natural inner product $\langle \cdot , \cdot \rangle_{(\Sigma,\mathscr{S})}$ is defined on $V_{(\Sigma,\mathscr{S})}$ by

$$\left\langle \bigotimes_{t \in \mathscr{S}} \alpha_t , \bigotimes_{t \in \mathscr{S}} \beta_t \right\rangle_{(\Sigma,\mathscr{S})} = \prod_{t \in \mathscr{S}} \langle \alpha_t , \beta_t \rangle_t , \tag{2.26}$$

where $\alpha_t, \beta_t \in V_t$ for $t \in \mathscr{S}$, and such that the direct sum in (2.25) is orthogonal. Thus (2.25) and (2.26) define a Hilbert space $V_{(\Sigma,\mathscr{S})}$, and we note that

$$V_{(\Sigma_1 \cup \Sigma_2, \mathscr{S}_1 \cup \mathscr{S}_2)} = V_{(\Sigma_1,\mathscr{S}_1)} \otimes V_{(\Sigma_2,\mathscr{S}_2)} , \tag{2.27}$$

where $(\Sigma_1 \cup \Sigma_2, \ \mathscr{S}_1 \cup \mathscr{S}_2)$ denotes the disjoint union of triangulated surfaces $(\Sigma_1, \mathscr{S}_1)$ and $(\Sigma_2, \mathscr{S}_2)$.

Next, let us consider the triangulated surface $(\Sigma^*, \mathscr{S}^*)$, i.e. $(\Sigma, \mathscr{S})$ with opposite orientation, but decorated with the same configuration of arrows and labels $i_l \in I$. For each oriented triangle t in $\mathscr{S}$ the oppositely oriented triangle t^* occurs in $\mathscr{S}^*$ and by inspection one sees that if $V_t = V_{ij}^{k^\vee}$, then $V_{t^*} = V_{i^\vee k^\vee}^{j}$.

There is a duality between these two spaces given by the bilinear form $(\cdot\,,\cdot)_{ij}^{k^\vee} : V_{ij}^{k^\vee} \times V_{i^\vee k^\vee}^{j} \to \mathbb{C}$ defined by

$$(\alpha, \beta)_{ij}^{k^\vee} 1_j = \beta \circ (1_{i^\vee} \underline{\otimes} \alpha) \circ (\psi_{i^\vee}^* \underline{\otimes} 1_j) \circ \varphi_j^* , \tag{2.28}$$

which can be shown (see ref. [10]) to be non-degenerate and symmetric, i.e.,

$$(\alpha, \beta)_{ij}^{k^\vee} = (\beta, \alpha)_{i^\vee k^\vee}^{j} , \tag{2.29}$$

and to be consistent with the identification (2.22), i.e.,

$$(\alpha, \beta)_{ij}^{k^\vee} = (\tilde{\alpha}, \underset{\sim}{\beta})_{ki}^{j^\vee} , \tag{2.30}$$

for $\alpha \in V_{ij}^{k^\vee}$, $\beta \in V_{i^\vee k^\vee}^{j}$. Hence the bilinear forms (2.28) define bilinear forms

$$(\cdot\,,\cdot)_t : V_t \times V_{t^*} \to \mathbb{C} , \tag{2.31}$$

yielding a duality between V_t and V_{t^*}, and we may define the bilinear form $(\cdot\,,\cdot)_{(\Sigma,\mathscr{S})} : V_{(\Sigma,\mathscr{S})} \times V_{(\Sigma^*,\mathscr{S}^*)} \to \mathbb{C}$ by

$$\left(\bigotimes_{t\in\mathscr{S}} \alpha_t , \bigotimes_{t\in\mathscr{S}} \beta_{t^*} \right)_{(\Sigma,\mathscr{S})} = \left(\prod_{l\in\mathscr{S}} F_{i_l} \right)^{-1} \prod_{t\in\mathscr{S}} (\alpha_t, \beta_{t^*})_t , \tag{2.32}$$

for $\alpha_t \in V_t$, $\beta_{t^*} \in V_{t^*}$, $t \in \mathscr{S}$, corresponding to a given labeling of links in $\mathscr{S}$. Requiring $(\cdot\,,\cdot)_{(\Sigma,\mathscr{S})}$ to be zero when acting on pairs of elements belonging to different direct summands in (2.25), i.e. corresponding to different labelings, (2.32) defines $(\cdot\,,\cdot)_{(\Sigma,\mathscr{S})}$ by bilinearity. Clearly, this form is non-degenerate and symmetric, i.e.,

$$(Z, Z')_{(\Sigma,\mathscr{S})} = (Z', Z)_{(\Sigma^*,\mathscr{S}^*)} , \tag{2.33}$$

for $Z \in V_{(\Sigma,\mathscr{S})}$, $Z' \in V_{(\Sigma^*,\mathscr{S}^*)}$, as a consequence of the corresponding properties of $(\cdot\,,\cdot)_t$, and yields a duality between $V_{(\Sigma,\mathscr{S})}$ and $V_{(\Sigma^*,\mathscr{S}^*)}$.

Defining the canonical antilinear isomorphism $Z \to Z^*$ from $V_{(\Sigma^*,\mathscr{S}^*)}$ onto $V_{(\Sigma,\mathscr{S})}$ as in section 1 it can be checked (see ref. [10]) that

$$Z^{**} = Z . \tag{2.34}$$

Note that the insertion of the factor $\prod_{l\in\mathscr{S}} F_{i_l}^{-1}$ in (2.32) is important for the validity of (2.34). Hence it follows that we can identify $V_{(\Sigma,\mathscr{S})}^*$ and $V_{(\Sigma^*,\mathscr{S}^*)}$ as Hilbert spaces.

This finishes our construction of the Hilbert spaces associated to triangulated surfaces.

2.3. PARTITION FUNCTIONS

Our next task will be to define the vectors $Z(M, \mathcal{T}) \in V_{(\partial M, \partial\mathcal{T})}$ associated to manifolds M equipped with a triangulation $\mathcal{T}$, which induces a triangulation $\partial\mathcal{T}$ on ∂M. The first step is to define $Z(\mathrm{B}^3, T_0) \in V(\mathrm{S}^2, \partial T_0)$, where (B^3, T_0) is an oriented three-ball triangulated by a single tetrahedron T_0 and $\mathrm{S}^2 = \partial\mathrm{B}^3$ is an oriented two-sphere. For this purpose we introduce the $6j$-symbols, which are linear mappings

$$F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} : V^i_{jp} \otimes V^p_{kl} \to V^i_{ql} \otimes V^q_{jk}\,, \tag{2.35}$$

for $i, j, k, l, p, q \in I$, defined by

$$\left\langle F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \alpha\otimes\beta, \gamma\otimes\delta \right\rangle 1_i = \alpha\circ(1_j \underline{\otimes} \beta)\circ(\delta^* \underline{\otimes} 1_l)\circ\gamma^*\,, \tag{2.36}$$

for $\alpha \in V^i_{jp}$, $\beta \in V^p_{kl}$, $\gamma \in V^i_{ql}$, $\delta \in V^q_{jk}$. Note that the right-hand side of (2.36) commutes with i and hence is proportional to 1_i, and the inner product on the left-hand side is the tensor product of the inner products on V^i_{ql} and V^q_{jk}. We thus have

$$\begin{aligned} F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} &\in \mathrm{Hom}(V^i_{jp} \otimes V^p_{kl}, V^i_{ql} \otimes V^q_{jk}) \\ &= V^i_{ql} \otimes V^q_{jk} \otimes (V^i_{jp})^* \otimes (V^p_{kl})^* \\ &= V^i_{ql} \otimes V^q_{jk} \otimes V^p_{j^\vee i} \otimes V^l_{k^\vee p}\,, \end{aligned} \tag{2.37}$$

which by inspection is seen to be the space associated to $(\mathrm{S}^2, \partial T_0)$ with a configuration of arrows and labels on links in T_0 a indicated on fig. 1, and which is a subspace of $V_{(\mathrm{S}^2, \partial T_0)}$.

Corresponding to each of the twelve elements of the symmetry group of the tetrahedron T_0 we may in this way for the given configuration of arrows and labels on the links in T_0 construct a vector in $V_{(\mathrm{S}^2, \partial T_0)}$, still with the convention

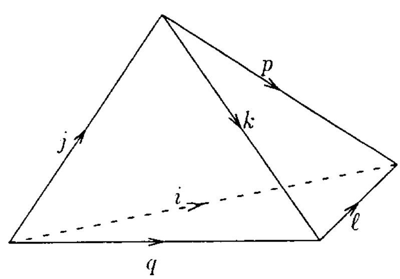

Fig. 1. A labeled tetrahedron.

that a flip of an arrow on a link l is compensated by a corresponding replacement $i_l \to i_l^\vee$. It turns out, however, that after suitable normalization these twelve vectors in $V_{(S^2,\partial T_0)}$ are in fact equal as a consequence of the following theorem [10].

Theorem 2.2. *For given $i, j, k, l, p, q \in I$ the vector*

$$W_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \equiv F_p F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \in V_{(S^2,\partial T_0)} \tag{2.38}$$

is invariant under the action of the tetrahedral symmetry group on the labels, and hence defines a unique vector in $V_{(S^2,\partial T_0)}$.

Sketch of proof. Since the tetrahedral symmetry group is generated by two of its elements, it is enough to prove the two identities,

$$F_p F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = F_{p^\vee} F_{p^\vee q^\vee}\begin{bmatrix} l & i^\vee \\ k^\vee & j \end{bmatrix}, \tag{2.39}$$

$$F_p F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = F_i F_{ik^\vee}\begin{bmatrix} q^\vee & j \\ l & p \end{bmatrix}, \tag{2.40}$$

in $V_{(S^2,\partial T_0)}$. This means that considered as elements in four-fold tensor products of intertwiner spaces the two sides of these equations are equal up to compositions with suitable tensor products of the isomorphisms $\alpha \to \tilde{\alpha}$.

Equation (2.39) is obtained by applying twice the identity

$$\left\langle F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha\otimes\beta), \gamma\otimes\delta \right\rangle = \left\langle \tilde{\beta}\otimes\tilde{\alpha}, F_{qp^\vee}\begin{bmatrix} i^\vee & j \\ l^\vee & k \end{bmatrix}(\tilde{\gamma}\otimes\delta) \right\rangle, \tag{2.41}$$

which in turn is obtained by a rather straightforward calculation using the definitions of $F_{pq}[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}]$ and the isomorphisms $\alpha \to \tilde{\alpha}$ (see ref. [10] for details).

The second identity (2.40) is obtained as a special case of the so-called pentagon identity for the 6*j*-symbols, which reads

$$\sum_{q \in I} F_{qm}^{(23)}\begin{bmatrix} n & j \\ r & k \end{bmatrix} F_{ir}^{(12)}\begin{bmatrix} n & q \\ u & l \end{bmatrix} F_{pq}^{(23)}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$

$$= P_{23} F_{pr}^{(13)}\begin{bmatrix} m & k \\ u & l \end{bmatrix} F_{im}^{(12)}\begin{bmatrix} n & j \\ u & p \end{bmatrix}, \tag{2.42}$$

as operators from $V_{ni}^u \otimes V_{jp}^i \otimes V_{kl}^p$ to $V_{rl}^u \otimes V_{mk}^r \otimes V_{nj}^m$, where the upper pair of indices in parenthesis indicates on which factors in the tensor product the 6*j*-symbol acts and P_{23} denotes permutation of the second and third factor. This identity is a well-known consequence of associativity of the tensor product $\otimes$ (see, e.g., refs. [5,10]). □

As a consequence of theorem 2.2 we may define

$$W(T_0) = \sum_{i,j,k,l,p,q \in I} W_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \in V_{(S^2, \partial T_0)} . \qquad (2.43)$$

Clearly, $W(T_0)$ is independent of which configuration of arrows on the links in T_0 is chosen. Up to a normalization factor F^2, where

$$F = \sum_{i \in I} F_i^{-2} , \qquad (2.44)$$

$W(T_0)$ equals $Z(B^3, T_0)$ as we shall see below.

In order to define $Z(M, \mathscr{T})$ for general triangulated manifolds $(M, \mathscr{T})$, we first distribute arrows and labels from I on the links in $\mathscr{T}$ and consider a labeled interior triangle t in $\mathscr{T}$ shared by the two tetrahedra T and T' and assume that t is oriented such that $t \in \partial T$ and $t^* \in \partial T'$. For the given labeling of links in $\mathscr{T}$ consider the corresponding components of $W(T)$ and $W(T')$. Denoting the labeled triangles in T and T' by t, t_1, t_2, t_3 and t^*, t'_1, t'_2, t'_3, respectively, these vectors belong to the subspaces $V_t \otimes V_{t_1} \otimes V_{t_2} \otimes V_{t_3}$ and $V_{t^*} \otimes V_{t'_1} \otimes V_{t'_2} \otimes V_{t'_3}$ of $V_{(S^2, \partial T)}$ and $V_{(S^{2*}, \partial T')}$, respectively. Hence they can be contracted by the bilinear form $(\cdot\,,\,\cdot)_t$ to a vector in $V_{t_1} \otimes V_{t_2} \otimes V_{t_3} \otimes V_{t'_1} \otimes V_{t'_2} \otimes V_{t'_3}$, which is a subspace of the Hilbert space associated to the triangulated manifold consisting of the two tetrahedra T and T' glued along t. Clearly, this procedure may be successively carried out for each interior triangle in $\mathscr{T}$, yielding a vector in $V_{(\partial M, \partial \mathscr{T})}$ after summing over all labelings. In order to ensure property (ii) in section 1, it is, however, necessary to include a factor $F_{i_l}^{-1}$ for each interior link l labeled by $i_l \in I$ because of eq. (2.32). This leads to the following general expression for $Z(M, \mathscr{T})$:

$$Z(M, \mathscr{T}) = F^{-|\mathscr{T}| - |\partial \mathscr{T}|/2} \sum_{(i_l)} \Bigg(\prod_{l \in \mathscr{T} \setminus \partial T} F_{i_l}^{-1} \Bigg) \Bigg(\bigotimes_{T \in \mathscr{T}} W(T) \Bigg)_{\mathrm{int}\, \mathscr{T}} , \qquad (2.45)$$

where $|\mathscr{T}|$ and $|\partial \mathscr{T}|$ denote the number of vertices in $\mathscr{T}$ and $\partial \mathscr{T}$, respectively, the summation is over all labelings of links, the tensor product is over all tetrahedra T in $\mathscr{T}$, and $(\cdot)_{\mathrm{int}\, \mathscr{T}}$ indicates the contractions associated with interior triangles. The weight factors F^{-1} and $F^{-1/2}$ corresponding to interior and boundary vertices, respectively, are necessary for the independence of $Z(M, \mathscr{T})$ on $\mathscr{T}$.

2.4. VERIFICATION OF AXIOMS AND TRIANGULATION INDEPENDENCE

Given an orientation preserving diffeomorphism $f: \Sigma \to \Sigma'$ between surfaces and a labeled triangulation $\mathscr{S}$ of Σ, the image $\mathscr{S}' = f(\mathscr{S})$ of $\mathscr{S}$ under f is in an obvious way a labeled triangulation of Σ'. Thus, if $\alpha_t \in V_t$ for each labeled triangle $t \in \mathscr{S}$, the vector $\otimes_{t \in \mathscr{S}} \alpha_t$ defines a vector both in $V_{(\Sigma, \mathscr{S})}$ and in $V_{(\Sigma', \mathscr{S}')}$, yielding a mapping $U_{\Sigma, \mathscr{S}}(f): V_{(\Sigma, \mathscr{S})} \to V_{(\Sigma', \mathscr{S}')}$, which may be written

$$U_{\Sigma,\mathscr{S}}(f)\left(\bigotimes_{t\in\mathscr{S}}\alpha_t\right)=\bigotimes_{t\in f(\mathscr{S})}\alpha_t\,, \tag{2.46}$$

and extended by linearity to all of $V_{(\Sigma,\mathscr{S})}$. Clearly, $U_{\Sigma,\mathscr{S}}(f)$ is unitary. Furthermore, if $F:M\to M'$ is an orientation preserving diffeomorphism between manifolds and $f=F|_{\partial M}:\partial M\to\partial M'$ denotes its restriction to the boundary, then for any triangulation $\mathscr{T}$ of M we have

$$Z(M',F(\mathscr{T}))=U_{\partial M,\partial\mathscr{T}}(f)Z(M,\mathscr{T}) \tag{2.47}$$

by (2.45), (2.46).

This establishes the triangulated version of property (i) in section 1. Property (ii) is a direct consequence of the definition (2.45) and (2.32) as mentioned above, whereas property (iii) is a consequence of eq. (2.41), which by inspection can be seen to be equivalent to

$$Z(\mathrm{B}^{3*},T_0^*)=Z(\mathrm{B}^3,T_0)^*\,, \tag{2.48}$$

with notation as above. The general case

$$Z(M^*,\mathscr{T}^*)=Z(M,\mathscr{T})^* \tag{2.49}$$

then follows easily by an inductive argument.

Property (v) is fulfilled by definition, whereas the triangulated version of (iv) is not fulfilled, but will be satisfied after removing the triangulation dependence as follows.

First one proves as in refs. [8,10] that $Z(M,\mathscr{T})$ only depends on M and $\partial\mathscr{T}$. One then defines for a surface Σ and triangulations $\mathscr{S}_1$ and $\mathscr{S}_2$ of Σ the mapping $h_{\mathscr{S}_2,\mathscr{S}_1}(\Sigma):V_{(\Sigma,\mathscr{S}_1)}\to V_{(\Sigma,\mathscr{S}_2)}$ by

$$\begin{aligned}h_{\mathscr{S}_2,\mathscr{S}_1}(\Sigma)&=Z([0,1]\times\Sigma,\mathscr{T})\in V^*_{(\Sigma,\mathscr{S}_1)}\otimes V_{(\Sigma,\mathscr{S}_2)}\\&=\mathrm{Hom}(V_{(\Sigma,\mathscr{S}_1)},V_{(\Sigma,\mathscr{S}_2)})\,,\end{aligned}$$

where $\mathscr{T}$ is a triangulation of $[0,1]\times\Sigma$ such that $\partial\mathscr{T}=\mathscr{S}_1^*\cup\mathscr{S}_2$. Since $h_{\mathscr{S}_2,\mathscr{S}_1}(\Sigma)$ is independent of the choice of $\mathscr{T}$ one easily shows that $h_{\mathscr{S},\mathscr{S}}(\Sigma)$ is a projection, and that the supports $V'_{(\Sigma,\mathscr{S})}$ of these projections can be consistently identified via the mappings $h_{\mathscr{S}_2,\mathscr{S}_1}(\Sigma)$ yielding a vector space V_Σ, such that the vectors $Z(M,\mathscr{T})$ are identified with a unique vector $Z(M)\in V'_{\partial M}$. Moreover, the inner products $\langle\cdot,\cdot\rangle_{(\Sigma,\mathscr{S})}$, the bilinear forms $(\cdot,\cdot)_{(\Sigma,\mathscr{S})}$ and the mappings $U_{(\Sigma,\mathscr{S})}(f)$ define unique corresponding ones $\langle\cdot,\cdot\rangle_\Sigma$, $(\cdot,\cdot)_\Sigma$ and $U_\Sigma(f)$ such that all the conditions (i)–(v) in section 1 are fulfilled.

This finishes our general construction of topological quantum field theories.

Remark 2.3. In the quasi-associative case the $6j$-symbols are defined by

$$\left\langle F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha\otimes\beta),\gamma\otimes\delta\right\rangle\cdot 1_i$$

$$=\alpha\circ(1_j\underline{\otimes}\beta)\circ\phi(j,k,l)\circ(\delta^*\underline{\otimes}1_l)\circ\gamma^*, \qquad (2.36')$$

as an operator from $V^i_{jp}\otimes V^p_{kl}$ to $V^i_{ql}\otimes V^q_{jk}$. Likewise the definition of the bilinear form $(\cdot\,,\cdot)^{k^\vee}_{ij}$ is modified to

$$(\alpha,\beta)^{k^\vee}_{ij}\cdot 1_j=\beta\circ(1_{i^\vee}\underline{\otimes}\alpha)\circ\phi(i^\vee,i,j)\circ(\psi^*_{i^\vee}\underline{\otimes}1_j)\circ\varphi^*_j, \qquad (2.28')$$

for $\alpha\in V^{k^\vee}_{ij}$, $\beta\in V^j_{i^\vee k^\vee}$, and the definition of the isomorphisms $\alpha\to\tilde\alpha$ becomes

$$\psi_k\circ(1_k\underline{\otimes}\alpha)\circ\phi(k,i,j)=\psi_{j^\vee}\circ(\tilde\alpha\underline{\otimes}1_j), \qquad (2.23')$$

for $\alpha\in V^{k^\vee}_{ij}$, $\tilde\alpha\in V^{j^\vee}_{ki}$.

The proof of the results leading to theorem 2.2 are then essentially identical to those in the associative case.

Remark 2.4. The simplest triangulation of the three-sphere S^3 is obtained by gluing two tetrahedra together along all four pairs of triangles. Using this triangulation one finds

$$Z(S^3)=1/F. \qquad (2.50)$$

Using a triangulation of $S^1\times S^2$ with six tetrahedra obtained by gluing together two identical prisms, each triangulated by three tetrahedra, one obtains

$$Z(S^1\times S^2)=1,$$

provided the two identities

$$N^k_{ij}=N^k_{ji}, \qquad \sum_{k\in I}F_k^{-1}N^k_{ij}=(F_iF_j)^{-1}, \qquad (2.51)$$

hold for $i, j, k\in I$, which will turn out to be the case in the examples discussed in sections 3 and 4.

3. The case of finite groups

3.1. FUNCTION ALGEBRAS

The simplest case in which our assumptions (1)–(5) in section 2 may be implemented is presumably when $\mathfrak{A}$ is the function algebra over a finite group G, i.e., $\mathfrak{A}=\mathcal{F}(G)=\{f:G\to\mathbb{C}\}$ with pointwise addition and multiplication and with complex conjugation as involution. $\mathcal{F}(G)$ is a commutative finite-dimensional algebra over $\mathbb{C}$, whose irreducible $*$-representations are one dimensional and labeled by the elements in G:

$$\pi_g(f)=f(g)\ ,\quad f\in\mathscr{F}(G), g\in G\ ,$$

where the right-hand side means multiplication by $f(g)$ in $\mathbb{C}$ equipped with the standard inner product.

Thus, identifying π_g with $g\in G$ we choose $I=G=\mathscr{C}$, $H_i=\mathbb{C}$ for $i\in G$ and set

$$i\underline{\otimes}j=ij\ ,\qquad i^\vee=i^{-1}\ ,$$

for $i, j\in G$.

Setting $u\underline{\otimes}v=u\otimes v$, for given intertwiners u and v between representations in I, the assumptions (1)–(4) in section 2 are trivially satisfied with $0=e$, where e is the identity in G, and

$$N_{ij}^k=\delta_{ij,k}=\begin{cases}1 & \text{if } ij=k\ ,\\ 0 & \text{otherwise}\ .\end{cases} \tag{3.1}$$

Choosing

$$\varphi_i,\ {}_i\varphi,\ \psi_i:\mathbb{C}\otimes\mathbb{C}\to\mathbb{C}$$

to be the canonical identification $x\otimes y\to xy$ it follows easily that the non-zero intertwiner spaces V_{ij}^k, $ij=k$, can be canonically identified with $\mathbb{C}$ as Hilbert spaces and that the bilinear forms $(\cdot\,,\,\cdot)_{ij}^k$ reduce to the canonical form $(\cdot\,,\,\cdot):\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ given by $(x, y)=xy$. Moreover, the technical condition (5) is then trivially satisfied with

$$F_i=1\ , \tag{3.2}$$

$$F_{pq}\begin{bmatrix} j & k\\ i & l\end{bmatrix}=\delta_{kl,p}\delta_{jk,q}\delta_{jp,i}\delta_{ql,i}\ . \tag{3.3}$$

In the definition (2.45) of $Z(M, \mathscr{T})$ the sum in this case is over all labelings of links in $\mathscr{T}$ by elements in G, and (3.3) shows that only configurations, for which the ordered product of labels around each triangle in $\mathscr{T}$ equals $e\in G$, contribute to the sum. Noting that by (2.44) and (3.2)

$$F=|G|\ , \tag{3.4}$$

we get for a closed manifold M that

$$Z(M)=Z(M,\mathscr{T})=|G|^{-|\mathscr{T}|}|V(G,\mathscr{T})|\ , \tag{3.5}$$

where $|V(G, \mathscr{T})|$ denotes the number of elements in the set $V(G, \mathscr{T})$ of mappings from the set of links in $\mathscr{T}$ to G fulfilling the constraint that the ordered product of elements around each triangle equals e (assuming some fixed configuration of arrows and replacing in the product the label g on a link by g^{-1} if the arrow points in negative direction).

This model was introduced by Dijkgraaf and Witten in ref. [12]. In fact, they introduced a class of models depending on a choice of three-cocycle

$$\omega: G\times G\times G \to \mathrm{U}(1)$$

for the group G, and the model just discussed corresponds to the trivial cocycle. The models corresponding to general cocycles ω are obtained similarly in our framework by regarding the previous setup in a quasi-associative setting with associativity isomorphisms $\phi(i, j, k)$ given by

$$\phi(i,j,k)=\omega(i,j,k)\ , \quad i,j,k\in G\ . \tag{3.6}$$

Since $\omega(i, j, k)\in \mathrm{U}(1)$ it is trivially a unitary intertwiner between $(\underline{i}\otimes\underline{j})\otimes\underline{k}$ and $\underline{i}\otimes(\underline{j}\otimes\underline{k})$ and the pentagon identity (2.1′) for $\pi_1, \pi_2, \pi_3, \pi_4\in G\equiv \mathscr{C}$ reduces to the cocycle condition

$$\omega(i,j,k)\omega(j,k,l)\omega(i,jk,l)=\omega(ij,k,l)\omega(i,j,kl)\ . \tag{3.7}$$

Equation (2.10′) is trivially satisfied in this case. Let us assume that ω is such that

$$\omega(i,j,k)=1 \quad \text{if } e\in\{i,j,k\}\ , \tag{3.8}$$

$$\omega(i,i^{-1},j)=1=\omega(i,j,j^{-1}) \quad \text{for } i,j\in G\ . \tag{3.9}$$

Then, letting φ_i, ${}_i\varphi$ and ψ_i be as above, it follows that the intertwiner spaces V^k_{ij}, $ij=k$, may still be canonically identified with $\mathbb{C}$ with the standard duality relation, and one can easily verify that the quasi-associative version of properties (1)–(5) in section 2 are satisfied with

$$F_i=1\ , \qquad F=|G|\ ,$$

$$F_{pq}\begin{bmatrix} j & k\\ i & l\end{bmatrix}=\delta_{kl,p}\delta_{jk,q}\delta_{jp,i}\delta_{ql,i}\omega(j,k,l)\ . \tag{3.10}$$

The only not entirely obvious property is the symmetry relation (5iv), which requires

$$\omega(i,j,(ij)^{-1})=\omega(j^{-1},i^{-1},ij)\ ,$$

and which is a rather simple consequence of (3.7)–(3.9).

Again it follows from (3.10) that only configurations $(i_l)_{l\in\mathscr{T}}$ in $V(G,\mathscr{T})$ contribute to $Z(M,\mathscr{T})$ and for such a configuration the weight of each labeled tetrahedron T equals $\omega(i,j,k)$, where i, j, k are the labels on the three links in a fixed non-closed path on T ordered according to the orientation of T as can be read off from fig. 1. Denoting the labels on the links in the chosen ordered path in T by $i_1(T)$, $i_2(T)$, $i_3(T)$, taking into account the configuration of arrows, we thus get for a closed manifold M,

$$Z(M)=Z(M,\mathscr{T})$$
$$=|G|^{-|\mathscr{T}|}\sum_{(il)_{l\in\mathscr{T}}\in V(G,\mathscr{T})}\prod_{T\in\mathscr{T}}\omega(i_i(T),i_2(T),i_3(T))\ . \tag{3.11}$$

In ref. [12] the partition functions for some manifolds M were calculated, as well as the dimensions of the vector spaces V_Σ. We shall not discuss these examples further here, but note that they illustrate nicely the fact that different models may be constructed for the same algebra corresponding to different choices of the associativity maps $\phi(i,j,k)$. Alternatively, the cocycles may be regarded as defining different quasi-Hopf algebra structures on the same underlying Hopf algebra $\mathscr{F}(G)$ as noted by G. Felder (private communication).

3.2. GROUP ALGEBRAS

A construction related to the one discussed above is based on the group algebra $\mathbb{C}[G]$ of a finite group G. This algebra may conveniently be considered as the complex vector space with basis G and with multiplication defined by extending the multiplication on G bilinearly, and involution given by the conjugate linear extension of the inversion $g \to g^{-1}$ on G. In fact, this algebra is also a Hopf algebra with comultiplication

$$\Delta g = g \otimes g\,, \quad g \in G\,,$$

and antipode

$$S(g) = g^{-1}\,, \quad g \in G\,.$$

As is well known the finite-dimensional $*$-representations of $\mathbb{C}[G]$ are obtained as extensions of finite-dimensional unitary representations of G, and they are completely reducible. Thus, we choose for I a set of representatives for each equivalence class of irreducible representations of G and we define the tensor product $\pi \underline{\otimes} \rho$ for two finite-dimensional representations π and ρ of $\mathbb{C}[G]$ in the usual way by setting $H_{\pi\underline{\otimes}\rho} = H_\pi \underline{\otimes} H_\rho$ as Hilbert spaces and

$$\pi \underline{\otimes} \rho(g) = \pi \otimes \rho(\Delta(g)) = \pi(g) \otimes \rho(g)\,, \quad g \in G\,.$$

Furthermore, the representative 0 in I of the trivial representation is assumed to be given by

$$H_0 = \mathbb{C}\,, \qquad 0(g) = 1 \quad \text{for } g \in G\,.$$

Defining the dual representation π' on $H_{\pi'} = H_\pi^*$ by

$$\pi'(g) = \pi(S(g))^{\mathrm{t}} = \pi(g^{-1})^{\mathrm{t}}\,, \quad g \in G\,,$$

where A^{t} denotes the dual operator to A, it is well known, and easy to show, that

(1) if π is irreducible then so is π',

(2) $\pi \otimes \rho$ contains the trivial representation if and only if ρ is equivalent to π' and in this case it contains 0 with multiplicity 1,

(3) $\pi'' = \pi$ if H_π^{**} is identified with H_π.

Using this we may define $i^\vee$, for $i \in I$, to be the representative of the equivalence class of i' in I. The properties (1)–(4) in section 2 then follow immediately, if one defines the tensor product $u\underline{\otimes}v$ of intertwiners to be the ordinary tensor product of operators.

As $\varphi_i:\mathbb{C}\otimes H_i\to H_i$ and ${}_i\varphi:H_i\otimes\mathbb{C}\to H_i$ we choose the canonical identifications given by

$$\varphi_i(1\otimes x)=x={}_i\varphi(x\otimes 1)\,,\quad x\in H_i\,,$$

implying the validity of (5i)–(5iii).

The choice of $\psi_k:H_k\otimes H_{k^\vee}\to\mathbb{C}$ requires some more care. Let for each $i\in I$ the mappings

$$\psi_i':H_i^*\otimes H_i\to\mathbb{C}\,,\qquad \psi_i'':H_i\otimes H_i^*\to\mathbb{C}$$

be given by

$$\omega_i\psi_i'(f\otimes x)=(f,x)=\psi_i''(x\otimes f)\,, \tag{3.12}$$

for $x\in H_i$, $f\in H_i^*$, where $(\cdot\,,\cdot)$ denotes the canonical bilinear form $H_i^*\times H_i\to\mathbb{C}$, and where ω_i is a sign to be determined below. If $i\neq i^\vee$, choose unitary intertwiners

$$u_i:H_{i^\vee}\to H_i^*\,,\qquad u_{i^\vee}:H_i\to H_{i^\vee}^*\,,$$

such that

$$\psi_i'\circ(u_i\otimes 1_i)=\psi_{i^\vee}''\circ(1_{i^\vee}\otimes u_{i^\vee})\,. \tag{3.13}$$

Since u_i and $u_{i^\vee}$ are both determined up to a phase and both sides of eq. (3.13) are intertwiners between $i^\vee\underline{\otimes}i$ and 0, and the latter occurs in $i^\vee\underline{\otimes}i$ with multiplicity 1, it follows that the phases can be chosen such that (3.13) is fulfilled, provided $i\neq i^\vee$. An easy calculation shows that (3.13) is also fulfilled with $i^\vee$ replacing i, if $\omega_{i^\vee}=\omega_i$.

If, on the other hand, $i=i^\vee$, then $u_i=u_{i^\vee}$ and we cannot exploit the arbitrary phase in order to ensure the validity of (3.13). Using the definition of ψ_i' and ψ_i'' and the fact that 0 occurs with multiplicity 1 in $i\underline{\otimes}i$ it is, however, straightforward to verify that

$$\omega_i\psi_i'\circ(u_i\otimes 1_i)=\pm\psi_i''\circ(1_i\otimes u_i)\,, \tag{3.14}$$

where the + sign occurs if 0 is contained in the symmetric part of $i\underline{\otimes}i$ with respect to the flip $x\otimes y\to y\otimes x$ on $H_i\otimes H_i$, and the − sign occurs otherwise. It is well known (see, e.g., ref. [13]) that this sign factor equals the so-called Frobenius–Schur index of i, denoted by η_i, and that

$$\eta_i=\begin{cases}1 & \text{if } i \text{ is a real representation of } G\,,\\ -1 & \text{otherwise}\,,\end{cases}$$

for any self-dual representation i.

If the + sign holds in (3.14) for all $i=i^\vee\in I$ we set $\omega_i=1$ and

$$\psi_i = \psi'_i \circ (u_i \otimes 1_i) = \psi''_{i^\vee} \circ (1_{i^\vee} \otimes u_{i^\vee}) \tag{3.15}$$

for each $i \in I$ and an easy calculation then shows that properties (5iv) and (5v) are fulfilled with

$$F_i^{-1} = \dim H_i \tag{3.16}$$

for $i \in I$. More generally, one finds the following result.

Theorem 3.1. *Assume that $i \to \omega_i \in \{\pm 1\}$ is an extension of the Frobenius–Schur index η_i, to all of I, which fulfills*

$$\omega_i \omega_j \omega_k = 1 \quad \text{if } N_{ij}^k \neq 0 \,. \tag{3.17}$$

If ψ_i is defined by (3.15) for $i \in I$, then all properties (1)–(5) in section 2 are satisfied with

$$F_i^{-1} = \omega_i \dim H_i \,, \qquad F = \sum_{i \in I} (\dim H_i)^2 = |G| \,.$$

The proof follows by a straightforward calculation after substituting (3.15) into (2.16) and using (3.12) and (3.17), and is left to the reader.

We note that all *finite abelian groups* as well as all *groups whose self-dual representations are all real*, such as the symmetric groups, are covered by this theorem, since we may choose $\omega_i = 1$ for all $i \in I$.

An obvious question is whether there exists a duality transformation relating the two constructions outlined in this section. We do not have an answer to this question. A kind of hybrid of the two constructions should be obtainable by considering a suitable class of representations of the Hopf algebra $D_\omega = \mathcal{F}(G) \otimes \mathbb{C}[G]$, introduced in ref. [14], depending on a three-cocycle ω on G. This algebra was used in ref. [15] for a surgery construction (see ref. [4]) of a topological quantum field theory and it was conjectured to coincide with the Dijkgraaf–Witten model discussed above, for the same choice of three-cocycle ω.

4. The case of quantum groups

In this section we explain how deformations of the enveloping algebras of the classical finite-dimensional Lie algebras give rise to a realization of assumptions (1)–(5) in section 2 and hence to a topological quantum field theory.

Denoting the Cartan matrix for a classical simple Lie algebra $\mathcal{G}$ by $(a_{ij})_{1 \leqslant i,j \leqslant m}$, which we for simplicity assume is symmetric, the relations defining the deformed algebra $\mathfrak{A} = \mathcal{U}_q \mathcal{G}$ over $\mathbb{C}$ are given in terms of generators E_i, F_i, K_i, K_i^{-1}, $1 \leqslant i \leqslant m$, as follows:

$$K_iK_j=K_jK_i, \qquad K_iK_i^{-1}=K_i^{-1}K_i=1\,,$$
$$K_iE_jK_i^{-1}=q^{a_{ij}/2}E_j\,, \qquad K_iF_jK_i^{-1}=q^{-a_{ij}/2}F_j\,,$$
$$[E_i,F_j]=\delta_{ij}\frac{K_i^2-K_i^{-2}}{q-q^{-1}}\,,$$
$$\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_q E_i^{(1-a_{ij}-n)}E_jE_i^n=0\,, \quad i\neq j\,,$$
$$\sum_{n=0}^{1-a_{ij}}(-1)^n\begin{bmatrix}1-a_{ij}\\ n\end{bmatrix}_q F_i^{(1-a_{ij}-n)}F_jF_i^n=0\,, \quad i\neq j\,,$$

where

$$\begin{bmatrix}r\\ s\end{bmatrix}_q=\frac{[r]_q!}{[s]_q![r-s]_q!}\,, \qquad [r]_q!=\prod_{j=1}^{r}\frac{q^j-q^{-j}}{q-q^{-1}}\,,$$

for q a complex number $\neq \pm 1$ and r, s integers. In the limit $q\to 1$ one recovers the relations defining $\mathscr{G}$ with $K_i=q^{H_i/2}$.

$\mathfrak{A}$ has a Hopf algebra structure with comultiplication $\Delta:\mathfrak{A}\to\mathfrak{A}\otimes\mathfrak{A}$ given by

$$\Delta(K_i)=K_i\otimes K_i\,, \qquad \Delta(E_i)=K_i\otimes E_i+E_i\otimes K_i^{-1}\,,$$
$$\Delta(F_i)=K_i\otimes F_i+F_i\otimes K_i^{-1}\,, \tag{4.1}$$

and with antipode $S:\mathfrak{A}\to\mathfrak{A}$ given by

$$S(K_i)=K_i^{-1}\,, \qquad S(E_i)=-qE_i\,, \qquad S(F_i)=-q^{-1}F_i\,. \tag{4.2}$$

For $|q|=1$, $\mathfrak{A}$ is an associative $*$-algebra with involution $a\to a^*$ given by

$$K_i^*=K_i^{-1}, \qquad E_i^*=F_i\,. \tag{4.3}$$

It follows from (4.1) and (4.3) that

$$\Delta(a)^*=\Delta'(a^*) \tag{4.4}$$

for $a\in\mathfrak{A}$, where $(a\otimes b)^*=a^*\otimes b^*$ for a, $b\in\mathfrak{A}$ and Δ' denotes the opposite comultiplication obtained from Δ by setting

$$\Delta'(a)=P\circ\Delta(a)\,, \tag{4.5}$$

where $P:\mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{A}\otimes\mathfrak{A}$ is the flip given by $P(a\otimes b)=b\otimes a$.

There exists an invertible element $R\in\mathfrak{A}\otimes\mathfrak{A}$ such that

$$R\Delta(a)=\Delta'(a)R \quad \text{for } a\in\mathfrak{A}\,. \tag{4.6}$$

In fact, one can choose $R\in\mathfrak{A}_+\otimes\mathfrak{A}_-$, where $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are the subalgebras of $\mathfrak{A}$ generated by K_i, E_i, $i=1, ..., m$, and K_i, F_i, $i=1, ..., m$, respectively, and explicit formulae for R have been obtained (see, e.g., ref. [16]). Furthermore, $R\in\mathfrak{A}_+\otimes\mathfrak{A}_-$

is uniquely determined by (4.6) up to a constant factor, which can uniquely be fixed such that

$$(\Delta\otimes 1)R=R_{13}R_{23}\,,\qquad (1\otimes\Delta)R=R_{13}R_{12}\,, \tag{4.7}$$

as elements in $\mathfrak{A}\otimes\mathfrak{A}\otimes\mathfrak{A}$, where $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$ etc. (see ref. [17]).

From the uniqueness of $R\in\mathfrak{A}_+\otimes\mathfrak{A}_-$ it follows that

$$R^*=P(R^{-1})\,. \tag{4.8}$$

We shall also need the fact (see ref. [18]) that there exists an invertible central element $c\in\mathfrak{A}$ such that

$$R_{21}R_{12}\cdot\Delta(c)=c\otimes c \tag{4.9}$$

in $\mathfrak{A}\otimes\mathfrak{A}$, where $R_{12}=R$ and $R_{21}=P(R)$.

If q is not a root of unity the representation theory of $\mathfrak{A}$ is essentially the same as for $\mathscr{G}$ in the sense that the finite-dimensional representations of $\mathfrak{A}$ can be obtained as deformations of those of $\mathscr{G}$ (see refs. [19,20]). If q is a root of unity the situation is different and it is convenient to distinguish representations according to whether their q-dimension vanishes or not. The q-dimension of a finite-dimensional representation π of $\mathfrak{A}$ is defined as

$$\dim_q\pi=\operatorname{tr}\,\pi(K_{2\rho}^2)\,, \tag{4.10}$$

where ρ is half the sum of positive roots and

$$K_\beta=\prod_{i=1}^{m}K_i^{m_i} \tag{4.11}$$

if $\beta=\sum_{i=1}^m m_i\alpha_i$ is an element in the root lattice, and $\alpha_1, \ldots, \alpha_m$ are the positive simple roots.

If $q=e^{i\pi/l}$, where l is an integer bigger than the Coxeter number for $\mathscr{G}$ we consider the set of dominant weights λ for $\mathscr{G}$ fulfilling

$$\langle\lambda+\rho,\alpha^\vee\rangle<l\quad\text{for all positive roots }\alpha\,. \tag{4.12}$$

The corresponding irreducible highest weight representations of $\mathscr{G}$ can then be shown, as in refs. [19,20], to be deformable into irreducible inequivalent highest weight representations π_λ of $\mathfrak{A}$ with positive q-dimension, and constitute a maximal set of representations with these properties. It follows from Shapovalov's determinant formula that these representations are $*$-representations with respect to a positive definite inner product, which is unique up to a positive constant multiple (see ref. [21]). For this to hold the restriction on the values of q is important.

We now let I be a set of representatives for each of the equivalence classes of representations π_λ, where λ fulfills (4.12), and we choose for the trivial representation the representative 0 acting on $H_0=\mathbb{C}$ by

$$0(K_i)=1\,,\qquad 0(E_i)=0(F_i)=0\,,\quad i=1,...,m\,. \tag{4.13}$$

0 is also called the co-unit.

We have thus defined $(\mathfrak{A}, I)$ and proceed to establish properties (1)–(5) in section 2.

First, in order to define the involution $i \to i^\vee$ on I we define the dual representation π' to a finite-dimensional representation π of $\mathfrak{A}$ acting on $H_{\pi'}=H_\pi^*$ by

$$\pi'(a)=\pi(S(a))^{\mathrm{t}} \tag{4.14}$$

for $a\in\mathfrak{A}$. It is not hard to see that π'_λ is equivalent to a representation $\pi_{\lambda'}$ for a weight λ' fulfilling (4.12). Furthermore, π'' is equivalent to π, since there exists an invertible element $u\in\mathfrak{A}$ such that

$$S^2(a)=u^{-1}au\,, \tag{4.15}$$

for $a\in\mathfrak{A}$ (see ref. [22]). Using this we may define $i^\vee$, for $i\in I$, to be the representative in I of i'.

Next, we define the tensor product $\pi\underline{\otimes}\rho$ for $\pi,\rho\in\mathscr{C}$, the class of representations equivalent to finite direct sums of representations in I. First, define the representation $\pi\otimes\rho$ by

$$\pi\otimes\rho(a)=\pi\otimes\rho(\Delta(a))\,,\quad a\in\mathfrak{A}\,, \tag{4.16}$$

acting on $H_\pi\otimes H_\rho$. Since $\mathfrak{A}$ is not semisimple when q is a root of unity, it turns out that this representation is generally not completely reducible in general for $\pi,\rho\in I$. However, there exists (see refs. [23,24]) a unique decomposition

$$H_\pi\otimes H_\rho=Z_{\pi,\rho}\oplus H_{\pi\underline{\otimes}\rho}\,,$$

where $Z_{\pi,\rho}$ is a direct sum of indecomposable $\mathfrak{A}$-modules all of which have vanishing q-dimension and the restriction of $\pi\otimes\rho$ to $H_{\pi\underline{\otimes}\rho}$ is a direct sum of representations in I. Furthermore,

$$H_{(\pi\underline{\otimes}\rho)\underline{\otimes}\eta}=H_{\pi\underline{\otimes}(\rho\underline{\otimes}\eta)} \tag{4.17}$$

for arbitrary representations $\pi,\rho,\eta\in\mathscr{C}$.

Thus, letting $\pi\underline{\otimes}\rho$ denote the restriction of $\pi\otimes\rho$ to $H_{\pi\underline{\otimes}\rho}$ it follows from (4.17) and coassociativity

$$(\Delta\otimes 1)\circ\Delta=(1\otimes\Delta)\circ\Delta\,, \tag{4.18}$$

that properties (1) and (2) are fulfilled, provided we can show that $\pi\underline{\otimes}\rho$ is a $*$-representation with respect to an inner product on $H_{\pi\underline{\otimes}\rho}$. In order to accomplish this we first note that the restriction to $H_{\pi\underline{\otimes}\rho}$ of the standard inner product $\langle\cdot,\cdot\rangle$ on $H_\pi\otimes H_\rho$ inherited from H_π and H_ρ does not apply because of eq. (4.5), since $\Delta\neq\Delta'$. However, combining eqs. (4.6) and (4.5) it follows that $\pi\underline{\otimes}\rho$ is a $*$-representation with respect to the sesquilinear form

$$\langle x, y\rangle_R = \langle \pi\otimes\rho(R)X, Y\rangle\ , \quad x, y\in H_{\pi\underline{\otimes}\rho}\ . \tag{4.19}$$

Because of uniqueness up to constant factors of such a sesquilinear form on each irreducible subspace, it follows that the restriction of $\langle\cdot,\cdot\rangle_R$ to each irreducible component $H_k\subseteq G_{\pi\underline{\otimes}\rho}$ is proportional to the inner product on H_k. Thus in order to obtain an inner product from $\langle\cdot,\cdot\rangle_R$ it has to be multiplied by a suitable constant. This can be done as follows. Let $i, j, k\in I$ and assume that k is contained in $i\underline{\otimes}j$. Denoting the constant multiplying $\langle\cdot,\cdot\rangle_R$ on $H_k\subseteq H_{i\underline{\otimes}j}$ by c_{ij}^k the requirement is that

$$c_{ij}^k\langle x, y\rangle_R = \overline{c_{ij}^k\langle y, x\rangle_R}\ , \quad x, y\in H_k\ ,$$

or, by the definition of $\langle\cdot,\cdot\rangle_R$,

$$c_{ij}^i i\otimes j(R) = \overline{c_{ij}^k}(i\otimes j)(R^*) \quad \text{on } H_k\ . \tag{4.20}$$

Inserting (4.8) into (4.20) this yields

$$i\otimes j(R_{21}R_{12}) = \overline{c_{ij}^k}/c_{ij}^k\ . \tag{4.21}$$

Comparing eq. (4.21) to eq. (4.9) we see that since c belongs to the center of $\mathfrak{A}$ and hence acts as multiplication by a constant c_i on each H_i, $i\in I$, we have to choose c_{ij}^k such that

$$\overline{c_{ij}^k}/c_{ij}^k = c_j c_j/c_k\ .$$

Below we show that c_i, $i\in I$, is a phase. Hence we may set

$$c_{ij}^k = \sqrt{c_k}/\sqrt{c_i}\sqrt{c_j}\ , \tag{4.22}$$

where the square roots have to be chosen so as to ensure positivity of $c_{ij}^k\langle x, x\rangle_R$ for $x\in H_k\subseteq H_{i\underline{\otimes}j}$. Using that $\langle\cdot,\cdot\rangle_R$ equals $\langle\cdot,\cdot\rangle$ in the limit $q\to 1$ for fixed i, j, k, the square roots must be chosen as continuous functions of q (Re $q>0$) tending to 1 as $q\to 1$.

This defines the desired inner product on $H_{i\underline{\otimes}j}$ for $i, j\in I$ and hence also on $H_{\pi\underline{\otimes}\rho}$ for $\pi, \rho\in\mathscr{C}$. As a consequence of eq. (4.7) and the factorized form (4.22) of the factors c_{ij}^k, it follows easily that this inner product is compatible with associativity, i.e., it defines a unique inner product on $H_{\pi\underline{\otimes}\rho\underline{\otimes}\eta}$ for $\pi, \rho, \eta\in\mathscr{C}$.

Note that we have assumed above that the form $\langle\cdot,\cdot\rangle_R$ does not vanish identically on $H_k\subseteq H_{i\underline{\otimes}j}$, a fact which follows by continuity for fixed i, j, k and l large. We shall not discuss the general case further here.

Thus we have verified conditions (1) and (2) of section 2. As concerns condition (3) it is easy to check from the definition of Δ and the trivial representation 0 that $H_{i\underline{\otimes}0} = H_i\otimes\mathbb{C}$, $H_{0\underline{\otimes}i} = \mathbb{C}\otimes H_i$ and that $i\underline{\otimes}0 = i\otimes 0$ and $0\underline{\otimes}i = 0\otimes i$ are equivalent to i via the canonical identifications

$$\varphi_i : \lambda\otimes x\to\lambda x\ , \qquad {}_i\varphi : x\otimes\lambda\to\lambda x\ . \tag{4.23}$$

That 0 is contained in $i^\vee \otimes j$ if and only if $i=j$, and that $N^0_{i^\vee j}=1$ is seen in a similar way as in the group case: Consider irreducible representations π and ρ of $\mathfrak{A}$ and identify $H^*_\pi \otimes H_\rho$ with $\mathrm{Hom}(H_\pi, H_\rho)$. Then the action of the representation $\pi' \otimes \rho$ of $\mathfrak{A}\otimes\mathfrak{A}$ on $T\in\mathrm{Hom}(H_\pi, H_\rho)$ is given by

$$\pi'\otimes\rho(a\otimes b)T=\rho(b)T\pi(S(a)) .$$

Using this one finds by direct calculation inserting $\Delta(E_i)$, $\Delta(F_i)$ and $\Delta(K_i)$, respectively, for $a\otimes b$, that $\pi'\otimes\rho$ contains 0 with invariant vector $T_0\in\mathrm{Hom}(H_\pi, H_\rho)$ if and only if

$$\rho(a)T_0=T_0\pi(a) \quad \text{for } a\in\mathfrak{A} ,$$

i.e., T_0 is an intertwiner between π and ρ. For π and ρ irreducible this means that π and ρ are equivalent and that T_0 is uniquely determined up to a constant factor in $\mathbb{C}$. Moreover, since in this case $\langle T_0, T_0\rangle_R\neq 0$ as we shall see below, it follows that 0 is a direct summand in $\pi'\underline{\otimes}\rho$, since the orthogonal complement to T_0 with respect to $\langle\cdot,\cdot\rangle_R$ in $H^*_\pi\otimes H_\rho$ is invariant under $\pi'\underline{\otimes}\rho$. Thus it follows that 0 is contained in $\pi'\underline{\otimes}\rho$ if and only if π and ρ are equivalent and property (3) is proven.

Property (4) follows immediately if one defines $u\underline{\otimes}v$ as the restriction of $u\otimes v$ to $H_{\pi\underline{\otimes}\rho}$ for intertwiners $u: H_\pi\to H_{\pi_1}$ and $v: H_\rho\to H_{\rho_1}$ with $\pi, \pi_1, \rho, \rho_1\in\mathscr{C}$.

Finally, we establish properties (5i)–(5v). Of these (5i)–(5iii) are obvious from the definition (4.23) of φ_i and ${}_i\varphi$. Moreover, φ_i and ${}_i\varphi$ are unitary, since the inner products on $\mathbb{C}\otimes H_i$ and $H_i\otimes\mathbb{C}$ equal the standard inner products, as follows by noting that $0\otimes i(R)=i\otimes 0(R)=1\otimes 1$ by the explicit formula for R quoted, e.g., in ref. [16], and, as a consequence, $c_0=1$.

Property (5iv) is less trivial, but can be obtained in a similar way as in the group algebra case. First, we note that the element

$$u=\sum_j b_j S(a_j) , \tag{4.24}$$

where $a_j, b_j\in\mathfrak{A}$ are given by

$$R=\sum_j a_j\otimes b_j \tag{4.25}$$

fulfills eq. (4.15) (see ref. [22]), and that

$$c^2=uS(u) , \qquad S(c)=c \tag{4.26}$$

(see ref. [18]). We can then conclude that $i(u)$ and $i(c)=c_i$ are unitary for $i\in I$ as follows: Since the element

$$K\equiv K^2_{2\rho} \tag{4.27}$$

fulfills $S^2(a)=KaK^{-1}$ for $a\in\mathfrak{A}$, which is easily verified for the generators of $\mathfrak{A}$ by direct calculation, it follows that uK belongs to the center of $\mathfrak{A}$. On the other hand, it follows from the explicit formula for R [16] and from (4.24) that u acts on a highest weight vector in H_i, $i\in I$, as multiplication by a phase, if $|q|=1$. Since this is obviously also the case for K, it follows that $i(u)i(K)$ equals a phase. Since $i(K)$ is unitary we conclude that $i(u)$ is unitary and hence $i(c)=c_i$ is a phase by eq. (4.26). In fact we can conclude that

$$i(uc^{-1})=i(K^*) \tag{4.28}$$

by computing the norm $\|I\|_{i'\underline{\otimes} i}$ of the invariant vector $I=1_i\in H_{i'\underline{\otimes} i}\subseteq \mathrm{Hom}(H_i, H_i)$. We get by using (4.24)

$$\|I\|^2_{i'\otimes i}=\frac{1}{c_i}\langle i'\otimes i(R)I, I\rangle=\mathrm{tr}(i(uc^{-1}))\,.$$

Thus it follows that $\mathrm{tr}(i(uc^{-1}))\geqslant 0$. Since also $\mathrm{tr}(i(K^*))=\mathrm{tr}(i(K))=\dim_q i>0$ for $i\in I$, (4.28) follows, and

$$\|I\|^2_{i'\otimes i}=\dim_q i \quad \text{for } i\in I\,. \tag{4.29}$$

More generally, we get for $T\in H_{i'\underline{\otimes} i}$

$$\langle T, I\rangle_{i'\underline{\otimes} i}=\mathrm{tr}(i(K^*)T)\,, \tag{4.30}$$

where $\langle\cdot,\cdot\rangle_{i'\otimes i}$ denotes the inner product on $H_{i'\otimes i}$.

Using (4.15) and identifying $H_i\otimes H_{i'}$ with $\mathrm{Hom}(H_i^*, H_i^*)$ it follows that an invariant vector in $H_{i\underline{\otimes} i'}$ is $i(K^*)^{\mathrm{t}}$ and that

$$\|i(K^*)^{\mathrm{t}}\|^2_{i\underline{\otimes} i'}=\mathrm{tr}\, i(K)=\dim_q i\,, \tag{4.31}$$

$$\langle S, \pi(K^*)^{\mathrm{t}}\rangle_{i\underline{\otimes} i'}=\mathrm{tr}\, S\,, \tag{4.32}$$

for $S\in H_{i\underline{\otimes} i'}\subseteq \mathrm{Hom}(H_i^*, H_i^*)$.

We thus define the intertwiners

$$\psi_i':H_{i'\underline{\otimes} i}\to\mathbb{C}\,,\qquad \psi_i'':H_{i\underline{\otimes} i'}\to\mathbb{C}\,,$$

$$\psi_i'=\omega_i\frac{\mathrm{tr}(i(K^*)T)}{\dim_q i}\,,\qquad \psi_i'=\frac{\mathrm{tr}\, S}{\dim_q i}\,, \tag{4.33}$$

which are partial isometries by (4.29)–(4.33), ω_i being a sign to be fixed below. Their adjoints are readily calculated to

$$\psi_i'^*(\mu)=\mu\frac{\omega_i I}{\dim_q i}\,,\qquad \psi_i''^*(\mu)=\mu\frac{i(K^*)^{\mathrm{t}}}{\dim_q i}\,,$$

for $\mu\in\mathbb{C}$.

We now choose for $i\neq i^{\vee}$ unitary intertwiners $u_i:H_{i^{\vee}}\to H_i^*$ and $u_{i^{\vee}}:H_i\to H_{i^{\vee}}^*$ such that

$$\psi_i'\circ(u_i\underline{\otimes} 1_i)=\psi_{i^{\vee}}''\circ(1_{i^{\vee}}\underline{\otimes} u_{i^{\vee}})\,,$$

whereas for $i=i^{\vee}$ we find that

$$\omega_i \psi_i' \circ (u_i \otimes 1_i) = \pm \psi_i'' \circ (1_{i^\vee} \otimes u_i)$$

for every intertwiner $u_i : H_i \to H_i^*$ between i and i' by direct calculation. Here the sign on the right-hand side, which is a continuous and hence constant function of q, may be determined by letting $q \to 1$, thus reducing to the classical group case, in which it is again determined by whether the invariant vector in $H_i \otimes H_i$ is symmetric or antisymmetric under the flip $x \otimes y \to y \otimes x$. Fixing ω to be this sign, it is well known and rather easy to see [10] that one can choose

$$\omega_i = i(K_{2\rho}^l) \quad \text{for } i \in I .$$

Since ω_i clearly fulfills

$$\omega_i \omega_j \omega_k = 1 \quad \text{if } N_{ij}^k \neq 0 ,$$

we may proceed to define

$$\psi_i = \psi_i' \circ (u_i \otimes 1_i) = \psi_{i^\vee}'' \circ (1_{i^\vee} \otimes u_{i^\vee})$$

for $i \in I$, and a straightforward calculation leads to properties (5iv) and (5v) with

$$F_i = \omega_i \dim_q i .$$

This finishes our discussion of the quantum group case.

Let us finally remark that the SU(2) case was discussed in ref. [8], where also some explicit calculations were carried out. Further calculational techniques were developed for this case in refs. [25,26] and should be extendable also to the general case.

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